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AN OPTIMAL BOUND FOR EMBEDDING LINEAR SPACES INTO PROJECTIVE PLANES

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Linear spaces with $v > n^2 - \frac{1}{2}n + 1$ points, $b \leqslant n^2 + n + 1$ lines and not constant point degree are classified. It turns out that there is essentially one class of such linear spaces which are not near pencils and which can not be embedded into any projective plane of order n.

0. Introduction

A *linear space* is a pair $L = (\mathscr{P}, \mathscr{L})$ consisting of a set $\mathscr{P}$ of *points* and a set $\mathscr{L}$ of at least two subsets of $\mathscr{P}$ called *lines* with the properties that

– any two distinct points p and q are contained in a unique line pq, and
– every line contains at least two points.

The *degree* r_p (or k_L) of a point p (or a line L) is the number of lines containing p (or points contained in L, respectively). A *parallel class* of L is a set Π of lines such that every point is contained in a unique line of Π. We denote the number of points by v and the number of lines by b.

A linear space L is called a *near pencil*, if it has a line which contains all but one of the points of L.

Given two linear spaces $L = (\mathscr{P}, \mathscr{L})$ and $L' = (\mathscr{P}', \mathscr{L}')$, L is said to be *embedded* in L', if $\mathscr{P} \subseteq \mathscr{P}'$ and $\mathscr{L} = \{L \cap \mathscr{P} \mid L \in \mathscr{L}' \text{ and } |L \cap \mathscr{P}| \geqslant 2\}$. In this case, we usually identify every line of L with the corresponding line of L'.

Erdös, Mullin, Sós and Stinson [3] conjectured that every linear space with $n^2 - n + 2 \leqslant v \leqslant b \leqslant n^2 + n + 1$ which has a point of degree at least $n + 2$ is a near pencil. We introduce a class of linear spaces which shows that this conjecture does not hold. Let $n = m^2$ be a perfect square, and suppose there exists a projective plane P of order n which has a Baer-subplane $B = (B, \Pi)$ (i.e., B is a projective plane of order m which is embedded in P). Then $P - B$, the *complement of B in P*, is the linear space which is obtained from P by removing the points of B. It is easy to see that Π is a parallel class of $P - B$. Therefore, we get in an obvious way a new linear space from $P - B$, if we let the lines of Π intersect in an "infinite" point ∞. We call this linear space the *closed complement of B in P*. It has $n^2 + n + 1$ lines and $n^2 - m + 1$ points. Furthermore, the point ∞ has degree $n + m + 1$. If we remove some points other than ∞ from the closed complement of $\boldsymbol{B}$ in $\boldsymbol{P}$, we obtain other examples of linear spaces which have a point of degree at least $n+2$ and satisfy $n^2-n+2 \leqslant v \leqslant b \leqslant n^2+n+1$. We shall show that for $v > n^2 - \frac{1}{2}n + 1$ there are no other such linear spaces (cf. Theorem 3 below).

For each integer $n \geqslant 1$ we define $A(n)$ to be the least positive integer such that every linear space with $A(n) \leqslant v \leqslant b \leqslant n^2+n+1$ is a near pencil or is embeddable into a projective plane of order n. Trivially, $A(n) \leqslant n^2+n+2$. Moreover $A(n) > n^2-n+1$, if $n-1$ is the order of a projective plane, since a projective plane of order $n-1$ can not be embedded into a projective plane of order n. Much better bounds are known. For instance, the classification of restricted linear spaces by Totten [6] yields $A(n) \leqslant n^2+1$. De Witte [9] proved $A(n) \leqslant n^2$, and he conjectured [8] that $A(n) \leqslant n^2-n+2$ except for some small values of n. However, if n is the order of a projective plane $\boldsymbol{P}$ which contains a Baer-subplane $\boldsymbol{B}$, then $A(n) \geqslant n^2 - \sqrt{n} + 2$, since the closed complement of $\boldsymbol{B}$ in $\boldsymbol{P}$ has a point of degree $n + \sqrt{n} + 1$ and can therefore not be embedded into any projective plane of order n. We shall show that $A(n) = n^2 - \sqrt{n} + 2$ in this case, whereas $A(n) \leqslant n^2 - \frac{1}{6}n + 1$ in every other case. This will be a consequence of the following more general results.

Theorem 1. *Let* $\mathbb{L}$ *be a linear space with* $n^2-n+1 < v \leqslant b \leqslant n^2+n+1$ *for some integer* n. *If* $v \neq 8$, *then* $\mathbb{L}$ *is a near pencil or the degree of every line is at most* $n+1$.

Theorem 2. *Let* $\mathbb{L}$ *be a linear space with* $n^2 - \frac{2}{3}n + 1 < v \leqslant b \leqslant n^2+n+1$ *for some integer* n. *If* $\mathbb{L}$ *has a point of degree at most* n, *then* $\mathbb{L}$ *is a near pencil or* $\mathbb{L}$ *can be embedded into a projective plane of order* n.

Theorem 3. *Let* $\mathbb{L}$ *be a linear space with* $n^2 - \frac{1}{2}n + 1 < v \leqslant b \leqslant n^2+n+1$ *for some integer* n. *Suppose that* $\mathbb{L}$ *is not a near pencil and that some point of* $\mathbb{L}$ *has degree at least* $n+2$. *Then* n *is a perfect square and* $\mathbb{L}$ *can be embedded into the closed complement of a Baer-subplane in a projective plane of order* n. *In particular,* $\mathbb{L}$ *can not be embedded into a projective plane of order* n.

Using a result of [4], we obtain the following corollary.

Corollary 1. *Let* $\mathbb{L}$ *be a linear space with* $n^2 - \frac{1}{2}n + 1 < v \leqslant b \leqslant n^2+n+1$ *for some integer* n. *Then one of the following cases occurs*:

(1) $\mathbb{L}$ *is a near pencil*;
(2) $\mathbb{L}$ *can be embedded into a projective plane of order* n;
(3) n *is a perfect square, and* $\mathbb{L}$ *can be embedded into the closed complement of a Baer-subplane in a projective plane of order* n;
(4) $\mathbb{L}$ *has* n^2+n+1 *lines,* $v \leqslant n^2 - \frac{1}{6}n$ *points and every point of* $\mathbb{L}$ *has degree* $n+1$.

Corollary 2. *Let n be any positive integer. If n is the order of a projective plane which has a Baer-subplane, then* $A(n) = n^2 - \sqrt{n} + 2$. *In every other case* $A(n) \leq n^2 - \frac{1}{6}n + 1$.

Since a desarguesian projective plane of order $n = m^2$ always has a Baer-subplane (see e.g. [1, Satz 7.1.5]), we obtain an optimal bound for embedding linear spaces into projective planes.

Corollary 3. *If* $n = p^{2s}$ *for a prime p, then* $A(n) = n^2 - \sqrt{n} + 2$.

Corollary 4. $A(1) = A(2) = 1$, $A(3) = 9$ *and* $A(4) = 16$.

Although it was not our intention, the methods which will be developed to prove these theorems give us also a result concerning blocking sets. A *blocking set* of a projective plane P is a set $\mathcal{G}$ of points of P with the property that every line of P contains a point of $\mathcal{G}$ as well as a point which is not in $\mathcal{G}$. Bruen [2] showed that $|\mathcal{G}| \geq n + \sqrt{n} + 1$ for every blocking set $\mathcal{G}$ of a projective plane P of order n with equality if and only if $\mathcal{G}$ is the set of points of a Baer-subplane of P. It is therefore of interest to characterize blocking sets containing the points of a Baer-subplane.

Theorem 4. *Let P be a projective plane of order n, and suppose* $\mathcal{G}$ *is a blocking set of P with* $|\mathcal{G}| < \frac{3}{2}n + 1$. *Then* $\mathcal{G}$ *contains the points of some Baer-subplane B of P if and only if there exists a set* Π *of at least n + 2 lines such that every point outside* $\mathcal{G}$ *is contained in a unique line of* Π. *In this case,* Π *consists of the lines of B.*

1. Preliminary results

We shall make use of the following results. The first one is a special case of a result due to Vanstone [7].

Result 1. *Suppose a linear space L with constant point degree n + 1, v points and at most* $n^2 + n + 1$ *lines has a point which occurs in s lines of degree n. If* $v + s \geq n^2 \geq 4$, *then L can be embedded into a projective plane of order n.*

Result 2. *Let L be a linear space with* $n^2 - \frac{1}{6}n < v \leq b \leq n^2 + n + 1$ *for some integer* $n \geq 2$. *If every point of L has degree at most n + 1, then L can be embedded into a projective plane of order n.*

Proof. See [4]. □

The following result was proved by Stanton and Kalbfleisch [5], and its corollary was already used in [3]. Defining

$$f(k, v) = 1 + \frac{k^2(v-k)}{v-1}, \quad 2 \leq k \leq v,$$

it can be stated as follows:

Result 3. *Every finite linear space with maximal line degree k satisfies* $b \geq f(k, v)$.

Corollary. *Every finite linear space with maximal line degree k satisfies* $b \geq \min\{f(k_1, v), f(k_2, v)\}$ *for all integers* k_1 *and* k_2 *with* $2 \leq k_1 \leq k \leq k_2 \leq v-2$.

2. A characterization of small blocking sets containing a Baer-subplane

In this section, we shall prove three lemmas which will be used in the next sections to classify certain finite linear spaces. However, also Theorem 4 will be a consequence of Lemma 2.2.

Lemma 2.1. *Let P be a finite projective plane and denote by n the order of P. Suppose* $\mathcal{q}$ *is a set of points and* Π *is a set of lines with the properties that no line of* Π *is contained in* $\mathcal{q}$ *and that every point outside* $\mathcal{q}$ *lies on a unique line of* Π. *Then the lines of* Π *are confluent in P, if one of the following conditions is satisfied*:

(a) $|\mathcal{q}| \leq 2n-1$ *and* $|\Pi| < n + 1 + \sqrt{n}$.

(b) $|\mathcal{q}| \leq 2n+1$, $|\Pi| \leq n+1$, $n \neq 3$ *and every line of* Π *contains at least two points outside* $\mathcal{q}$.

Proof. We prove both parts together. Suppose from now on that (a) or (b) is fulfilled.

Let $\mathcal{p}$ be the set of points not in $\mathcal{q}$, and set $v = |\mathcal{p}|$. For every line L, we denote by k_L the number of points in $L \cap \mathcal{p}$ and call k_L the degree of L. Trivially,

$$n^2 - n \leq n^2 + n + 1 - |\mathcal{q}| = |\mathcal{p}| = v = \sum_{L \in \Pi} k_L. \tag{1}$$

If Π contains a line L of degree $n+1$, then obviously $\Pi = \{L\}$. If Π contains a line N of degree n, then every line of Π contains the unique point of $N \cap \mathcal{q}$. W.l.o.g. we may therefore assume that every line of Π has degree at most $n-1$.

Choose a line G of Π of maximal degree, and put $d = n + 1 - k_G \geq 2$ and $G \cap \mathcal{q} = \{p_1, \ldots, p_d\}$. Furthermore, denote by M_i the set of lines other than G of Π which contain p_i, and set $m_i = |M_i|$, $i = 1, \ldots, d$. W.l.o.g. we may assume that $m_i \geq m_j$ for $i < j$. Finally, set $M = M_2 \cup \cdots \cup M_d$, $m = |M|$ and $a = |\Pi| - n - 1$. Then

$$m_1 + m = \sum_{i=1}^{d} m_i = |\Pi| - 1 = n + a.$$

To prove Lemma 2.1, we have to show that $M = \emptyset$. We shall do this in several steps.

Step 1. If $i, j \in \{1, \ldots, d\}$ with $i \neq j$, then every line of M_i has degree at most $n - m_j$.

For: A line L of M_i intersects each of the lines of M_j in a point of $\mathscr{g}$. Since L also contains p_i, it follows $k_L \leq n - m_j$.

Step 2. If $a < 0$, then $M = \emptyset$.

For: Suppose $a < 0$. Then (1) shows $v = n^2 - n$, $|\Pi| = n$ and $k_L = n - 1$ for every line L of Π. Thus $|\mathscr{g}| = 2n + 1$ so that (a) is not fulfilled. Hence (b) is fulfilled, i.e., $n \neq 3$ and every line has degree at least two. In view of $k_G = n - 1$, we obtain $n \geq 4$.

Assume $M \neq \emptyset$, and let L be a line of M. Then $n - 1 = k_L \leq n - m_1$ by Step 1. This contradicts $m_1 + m_2 = |\Pi| - 1 = n - 1$, $m_1 \geq m_2$ and $n \geq 4$. Consequently $M = \emptyset$.

Step 3. M_1 contains a line of degree $n - 1$.

For: In view of $v > k_G + n(n-2)$, p_1 is contained in a line L other than G of degree at least $n - 1$.

Assume by way of contradiction that $L \notin M_1$. Then each of the points of $L \cap \mathscr{p}$ lies on a line of M. Thus $m \geq k_L \geq n - 1$. In view of $m_i \geq m_j$ for $i < j$, we obtain

$$m_1 \geq m_2 \geq \frac{m}{d-1} \geq \frac{n-1}{d-1}.$$

It follows

$$n + a = m_1 + m \geq \frac{n-1}{d-1} + n - 1.$$

i.e.,

$$n - 1 \leq (a+1)(d-1). \tag{2}$$

First consider the case $a \leq 0$. Then (2) shows $a = 0$ and $d \geq n$. Consequently, $k_X = k_G = n + 1 - d = 1$ for every line X of Π. Thus

$$v = |\Pi| = n + 1 \leq n^2 - n + 1.$$

This and $k_G = 1$ contradicts (a) as well as (b).

Now suppose $a \geq 1$. Then $|\Pi| \geq n + 2$ so that (a) is fulfilled. In particular $v \geq n^2 - n + 2$. Together with (1) and (2) follows

$$\begin{aligned} n^2 - (a+1)(d-1) \leq n^2 - n + 1 < v \\ \leq |\Pi|\, k_G = (n + 1 + a)(n + 1 - d) \\ = n^2 + n(a + 2 - d) - (a+1)(d-1). \end{aligned}$$

We obtain $d < a + 2$. (2) and $n > a^2$ yield $d = a + 1$. It follows $m_1 > a$, since

$$a^2 + a < n + a = \sum_{i=1}^{d} m_i \leqslant dm_1.$$

Now, $n + a = m_1 + m$ and $m \geqslant n - 1$ implies $m_1 = a + 1$ and $m = n - 1$. Because of $k_X \leqslant n - m_1$ for every line X of M (Step 1) and $k_X \leqslant k_G$ for every line X of M_1, (1) shows

$$n^2 - n + 2 \leqslant v = \sum_{X \in \Pi} k_X \leqslant (m_1 + 1)k_G + m(n - m_1) = n^2 + 1 - a(a + 1).$$

We get $a(a + 1) \leqslant n - 1$, and therefore

$$a(a + 1) \leqslant n - 1 = m \leqslant (d - 1)m_2 \leqslant (d - 1)m_1 = a(a + 1).$$

Together this implies $m_1 = m_2 = a + 1$, $a(a + 1) = n - 1$ and $k_X = k_G = n - a$ for every line X of M_1. This contradicts Step 1.

Consequently, L is a line of M_1. Since L has degree at least $n - 1$, it follows that L has degree $n - 1$.

Step 4. $M = \emptyset$.

For: In view of Step 2, we may assume $a \geqslant 0$. By (1),

$$n^2 - n \leqslant v \leqslant |\Pi|\, k_G = (n + 1 + a)(n + 1 - d) = n^2 - (d - a - 2)n - (d - 1)(a + 1).$$

Since $a \geqslant 0$, we obtain $d \leqslant a + 2$.

Let L be a line of degree $n - 1$ of M_1, and denote by p the unique point other than p_1 of $L \cap \mathscr{q}$. Obviously, every line of M contains p. In particular, $m \leqslant d - 1$. From $n + a = m_1 + m$ and $d \leqslant a + 2$ follows therefore $m_1 \geqslant n - 1$. Step 1 shows that every line of M has degree at most $n - m_1 \leqslant 1$. Thus, if (b) holds, then $M = \emptyset$. If (a) is fulfilled, then

$$n^2 - n + 2 \leqslant v = \sum_{X \in \Pi} k_X \leqslant k_G + m_1(n - 1) + m \cdot 1 \leqslant m_1(n - 1) + n.$$

so that $m_1 \geqslant n$ and hence $M = \emptyset$ as before.

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let P be a finite projective plane and denote by n the order of P. Suppose $\mathscr{q}$ is a set of points and Π is a set of lines with the properties that no line of Π is contained in $\mathscr{q}$ and every point outside $\mathscr{q}$ lies on a unique line of Π. Denote by B the set of points which lie on at least two of the lines of Π. If $|\mathscr{q}| < \frac{3}{2}n + 1$, then either B consists of just one point or (B, Π) is a Baer-subplane of P.*

Proof. In view of Lemma 2.1, we may assume that $|\Pi| = n + 1 + m$ for some integer m with $m^2 \geqslant n$. We have to show that (B, Π) is a Baer-subplane of P. We shall do this in several steps. As in the proof of Lemma 2.1, we denote by $\mathscr{p}$ the set of points not in $\mathscr{q}$, we set $v = |\mathscr{p}|$ and we define the degree $k_L = |L \cap \mathscr{p}|$ of L for every line L of P.

Step 1. Every line of P has degree at most n.
For: This follows from $|\Pi| > n+1$.

Step 2. If N is a line of degree n and if $N \cap \mathcal{q} = \{q\}$, then $N \notin \Pi$ and q is contained in exactly $m+1$ of the lines of Π.
For: Each of the n points of $N \cap \mathcal{p}$ is on a unique line of Π. Since q is not contained in every line of Π, this shows $N \notin \Pi$. Furthermore, each of the $m+1$ lines of Π which does not intersect N in a point of $\mathcal{p}$, contains q.

Step 3. Every line of Π contains at least $m+1$ points of B.
For: Let L be a line of Π, and denote by p a point of $L \cap \mathcal{p}$. Then

$$n^2 - \tfrac{1}{2}n < v = 1 + \sum_{p \in X} (k_X - 1).$$

Because every line has degree at most n, it follows that p lies on a line N of degree n. Step 2 shows $N \notin \Pi$. Let q be the point of $N \cap \mathcal{q}$. Then each of the lines of Π through q intersects L in a point of B. Therefore, Step 3 follows from Step 2.

Step 4. Every point of B is contained in exactly $m+1$ lines of Π.
For: Let q be a point of B, and denote by x the number of lines of Π which contain q. By the definition of B, $x \geqslant 2$. Assume by way of contradiction that $x \neq m+1$. Then every line which contains q has degree at most $n-1$ (see Step 1 and Step 2). Therefore, Step 3 shows

$$n^2 - \tfrac{1}{2}n < v \leqslant x(n-m) + (n+1-x)(n-1) = n^2 - 1 - x(m-1).$$

In view of $n \leqslant m^2$, we obtain $2x \leqslant m+1$. Since $v > (n+1)(n-2)$, we see furthermore that q is contained in a line L of degree $n-1$. Step 3 and $m+1 \geqslant \sqrt{n}+1 > 2$ show $L \notin \Pi$. Let $q' \neq q$ be the second point of $L \cap \mathcal{q}$, and denote by y the number of lines of Π which contain q'. Obviously, $x+y = |\Pi| - k_L = m+2$. Consequently, $y \geqslant 2$, i.e., q' is a point of B, and $y \neq m+1$. As before, we can show that $2y \leqslant m+1$. This contradiction proves Step 4.

Step 5. (B, Π) is a Baer-subplane of P.
For: Let L be a line of Π, and denote by x the number of points of B on L. By Step 4, $|\Pi| = 1 + xm$ so that $n + m = xm$. Step 3 shows that $x \geqslant m+1$. In view of $m^2 \geqslant n$, we obtain $x = m+1$ and $n = m^2$. Consequently, every line of Π contains exactly $m+1$ points of B. This and Step 4 implies that any two distinct points of B are on a unique line of Π. Thus, (B, Π) is a Baer-subplane of P. □

Proof of Theorem 4. Let $\mathcal{q}$ be a blocking set in a projective plane P of order n, and suppose $|\mathcal{q}| < \tfrac{3}{2}n + 1$. If $\mathcal{q}$ contains the set of points of a Baer-subplane

(B, Π), then obviously $|\Pi| = n + 1 + \sqrt{n}$ and every point outside B lies on a unique line of Π. On the other hand, if Π is a set of lines with $|\Pi| > n + 1$, and such that every point outside $\mathscr{q}$ lies on a unique line of Π, then Lemma 2.2 shows that $\mathscr{q}$ contains the set of points of a Baer-subplane of $\boldsymbol{P}$, since no line of Π is contained in the blocking set $\mathscr{q}$. □

Lemma 2.3. *Let* $\mathbf{L}$ *be a linear space with* $v \geq n^2 - n + 1$ *points and* $b \leq n^2 + n$ *lines for some integer* $n \geq 2$. *If every point of* $\mathbf{L}$ *has degree* $n + 1$, *then* $\mathbf{L}$ *can be embedded into a projective plane of order* n.

Proof. Let $s \in \{0, 1, \ldots, n+1\}$ be maximal with the property that there are mutually disjoint parallel classes $\Pi_1, \ldots, \Pi_s$ with $|\Pi_i| = n$.

Assume by way of contradiction that $s < n + 1$. Let $\mathscr{M}$ be the set of lines not contained in one of the parallel classes Π_i. Then $|\mathscr{M}| = b - sn \leq (n + 1 - s)n$. Counting the set

$$\{(p, L) \mid L \text{ is a line of } \mathscr{M} \text{ and } p \text{ is a point of } L\}$$

in two different ways, we get

$$v(n + 1 - s) = \sum_{L \in \mathscr{M}} k_L.$$

Since $|\mathscr{M}|\,(n - 1) \leq (n + 1 - s)n(n - 1) < v(n + 1 - s)$, this shows that some line N of $\mathscr{M}$ has degree at least n. Because every point of N has degree $n + 1$ and in view of $b \leq n^2 + n$, N has degree n. Obviously N is contained in a unique parallel class Π of $\boldsymbol{L}$. Since N has degree n, Π is disjoint to the parallel classes Π_i, and Π contains at most n lines. Because every line of $\boldsymbol{L}$ has degree at most n, we also have $|\Pi| \geq v/n$ and hence $|\Pi| = n$. This contradicts the choice of s.

Consequently $s = n + 1$. Remembering the way how an affine plane can be embedded into a projective plane, we see that $\boldsymbol{L}$ can be embedded into a linear space $\boldsymbol{L}'$ with $v + n + 1$ points, $b + 1$ lines and constant point degree $n + 1$. Result 1 shows that $\boldsymbol{L}'$ can be embedded into a projective plane $\boldsymbol{P}$ of order n. This proves Lemma 2.3, because $\boldsymbol{L}$ is embedded in $\boldsymbol{L}'$. □

3. The proofs of Theorem 1 and Theorem 2

We start with the proof of Theorem 1. Though this theorem can not be found in [3], we remark that our proof follows an idea of there (see [3, Lemma 3.4]).

Proof of Theorem 1. Let $\boldsymbol{L}$ be a linear space with maximal line degree $k \geq n + 2$ and $n^2 - n + 2 \leq v \leq b \leq n^2 + n + 1$ for some integer n. We may asssume that $\boldsymbol{L}$ is not a near pencil, i.e, that $k \leq v - 2$, and we have to show that $v = 8$.

Assume by way of contradiction that $n \geq 4$. By the corollary of Result 3, we have $b \geqslant \min\{f(n+2, v), f(v-2, v)\}$. In view of $v \geqslant n^2-n+2$ and $n \geqslant 4$, it is easy to see that

$$f(v-2, v) \geqslant f(n+2, v) \geqslant f(n+2, n^2-n+2)$$
$$= n^2+n+1+\frac{2n^3-4n^2-9n}{n^2-n+1} > n^2+n+1.$$

This contradicts $b \leqslant n^2+n+1$.

Consequently $n \leqslant 3$. By Result 3, $b \geqslant f(k, v)$. In view of $f(k, v) \geqslant f(k, k+2) > 2k-1 \geqslant 2n+3$, we obtain $n=3$ and $k \in \{5, 6\}$. Now, $8 \leqslant v \leqslant b \leqslant 13$ and $b \geqslant f(k, v)$ show $v=8$. $\square$

Remark. There are two linear spaces with $8=v<b \leqslant 13$ and maximal line degree $k \geqslant n+2=5$. One of them has one line of degree 5, three lines of degree 4 and nine lines of degree 2. The other one has one line of degree 6, one line of degree 3 and ten lines of degree 2. In particular, $A(3) \geqslant 9$.

For the rest of this section, $\boldsymbol{L}$ will denote a linear space satisfying the conditions of Theorem 2. Let n be the integer with $n^2-\frac{2}{3}n+1<v \leqslant b \leqslant n^2+n+1$, and denote by p_0 a point of degree at most n. In order to prove Theorem 2, we may assume that $\boldsymbol{L}$ is not a near pencil so that every line has degree at most $n+1$ by Theorem 1. We shall prove several lemmas for $\boldsymbol{L}$.

Lemma 3.1. *p_0 has degree n and every other point has degree at least $n+1$. Furthermore, $v \geqslant n^2-n+3$, p_0 is contained in at least two lines of degree $n+1$, and every line containing p_0 has degree at least three.*

Proof. Assume by way of contradiction that $v \leqslant n^2-n+2$. Then $n^2-\frac{2}{3}n+1< v \leqslant n^2-n+2$, and hence $n=2$ and $v=4$. Since p_0 has degree at most n, it follows that $\boldsymbol{L}$ is a near pencil, a contradiction.

Consequently, $v \geqslant n^2-n+3$. If c denotes the number of lines of degree $n+1$ containing p_0, then

$$n^2-n+2 \leqslant v-1 \leqslant cn+(r_{p_0}-c)(n-1)=r_{p_0}(n-1)+c.$$

This shows that p_0 has degree n and that $c \geqslant 2$. In view of $c \geqslant 2$, every point other than p_0 has degree at least $n+1$. Finally, if L is a line containing p_0, then

$$n^2-n+3 \leqslant v \leqslant k_L+(r_{p_0}-1)n=n^2-n+k_L$$

so that L has degree at least three. $\square$

Lemma 3.2. *If every point other than p_0 has degree $n+1$, then $\boldsymbol{L}$ can be embedded into a projective plane of order n.*

Proof. Suppose every point other than p_0 has degree $n+1$. Then a line of degree

$n+1$ which contains p_0 intersects every line. Thus $b=n^2+n$. Because every line containing p_0 has degree at least 3, we obtain a linear space L' with constant point degree $n+1$ from L, if we remove the point p_0. L' has $v-1 \geqslant n^2-n+2$ points and n^2+n lines. Lemma 2.3 shows that L' can be embedded into a projective plane P of order n. Let Π be the set of lines through p_0. Lemma 2.1(a) shows that the lines of Π are confluent in P (this follows also from the fact that Π contains a line of degree n of L'). Hence, L can be embedded into a projective plane isomorph to P. □

In order to prove Theorem 2, we may assume from now on that some point of L has degree at least $n+2$. This will lead to a contradiction, which then completes the proof of Theorem 2.

Lemma 3.3. *L has n^2+n+1 lines. Furthermore, $n \geqslant 4$ and there exists a line H with the following properties*:
(a) *H is parallel to every line of degree $n+1$*;
(b) *every point on H has degree $n+2$*;
(c) *every point outside H and other than p_0 has degree $n+1$.*

Proof. Let q be some point of degree at least $n+2$, and let L be a line of degree $n+1$ through p_0 which does not contain q (Lemma 3.1). Then q is contained in a line H parallel to L. Since every point other than p_0 of L has degree at least $n+1$, L intersects at least n^2+n-1 lines. In view of $b \leqslant n^2+n+1$, it follows $b=n^2+n+1$, L intersects exactly n^2+n-1 lines so that every point other than p_0 of L has degree $n+1$, and it follows that L is parallel to exactly one line. Because H is the unique line parallel to L, every point of H has degree $k_L+1=n+2$ and every point outside H and L has degree $k_L=n+1$. This proves (b) and (c).

Let G be any line of degree $n+1$. Obviously, G contains p_0, and some point outside G has degree $n+2$. As before, it follows that G does not contain a point of degree $n+2$. This shows that H is parallel to every line of degree $n+1$, proving (a).

Since p_0 is contained in two lines of degree $n+1$ (Lemma 3.1), (a) implies that p_0 has degree at least $k_H+2 \geqslant 4$. Thus, $n \geqslant 4$. □

Lemma 3.4. *Every line other than H contains at least two points of degree $n+1$.*

Proof. Let L be any line other than H. If L does not contain p_0, then L intersects every line of degree $n+1$ in a point of degree $n+1$ (Lemma 3.3) so that L contains at least two points of degree $n+1$ (Lemma 3.1). Therefore, we may assume $p_0 \in L$. Let G be a line other than L through p_0 which intersects H. By Lemma 3.3(a), G has degree at most n. It follows

$$n^2-\tfrac{2}{3}n+1<v \leqslant k_L+(k_G-1)+(r_{p_0}-2)n \leqslant n^2-n-1+k_L.$$

Since $n \geqslant 4$, we obtain $k_L \geqslant 4$. Lemma 3.3 shows therefore that L contains at least two points of degree $n+1$. □

In view of Lemma 3.3 and Lemma 3.4, we obtain a linear space L' with constant point degree $n+1$, if we remove p_0, H and every point of H from L. L' has n^2+n lines and $v' := v - 1 - k_H$ points. Furthermore,

Lemma 3.5. *L' has at least $n^2 - n$ points and L' is embeddable into a projective plane of order n.*

Proof. If $v' > n^2 - n$, then Lemma 3.5 follows from Lemma 2.3. W.l.o.g. we may therefore assume that $v' \leqslant n^2 - n$. This implies that $k_H > \frac{1}{3}n$.

Consider the lines p_0q, $q \in H$, and choose a line N with maximal degree among them. Then

$$n^2 - \tfrac{2}{3}n < v - 1 = \sum_{p_0 \in X} (k_X - 1) \leqslant (r_{p_0} - k_H)n + k_H(k_N - 1)$$
$$= n^2 - k_H(n + 1 - k_N) < n^2 - \tfrac{1}{3}n(n + 1 - k_N).$$

Consequently, $k_N \geqslant n$. Lemma 3.3(a) shows $k_N = n$.

Let Π be the set of lines which are parallel to N, and let M denote the set of lines of Π which intersect H. Obviusly, $|\Pi| = n$ (cf. Lemma 3.3), $|M| = 2(k_H - 1)$ and every line of Π has degree at most $r_{p_0} - 1 = n - 1$. If we set $d_X = n + 1 - k_X$ for every line X of Π, then it follows

$$v - n + (k_H - 1) = \sum_{p \notin N} (r_p - k_N) = \sum_{X \in \Pi} k_X = n(n-1) - \sum_{X \in \Pi} (d_X - 2)$$
$$\leqslant n(n-1) - \sum_{X \in M} (d_X - 2).$$

Let q be a point on H with $\{q\} \neq H \cap N$, and denote by X_1 and X_2 the two lines of M which contain q. Then X_i, $i = 1, 2$, is parallel to N and every line of $\Pi - \{X_1, X_2\}$. Thus there are at least $n - 1$ lines which are parallel to X_1 and X_2. On the other hand, Lemma 3.3 shows that the number of lines parallel to X_1 and X_2 is $d_{X_1}d_{X_2} - 1$. It follows $d_{X_1}d_{X_2} \geqslant n$; in particular $d_{X_1} + d_{X_2} \geqslant 2\sqrt{n}$. Consequently

$$\sum_{X \in M} (d_X - 2) \geqslant 2(k_H - 1)(\sqrt{n} - 2),$$

and thus

$$(k_H - 1)(2\sqrt{n} - 3) \leqslant n(n-1) - v + n < \tfrac{1}{3}(2n - 3). \qquad (*)$$

Assume $n \geqslant 9$. Then

$$k_H < 1 + \frac{2n-3}{9} = \frac{n+2}{3} - \frac{n}{9} < \tfrac{1}{3}n,$$

a contradiction. Consequently $4 \leqslant n \leqslant 8$. (*) implies $k_H < 3$. Thus $k_H = 2$ and $v' = v - 3 \geqslant n^2 - n$ (Lemma 3.1). If we remove p_0, H and one of the points of H from $\boldsymbol{L}$, then we obtain a linear space $\boldsymbol{L}''$ with $b - 1 = n^2 + n$ lines, $v - 2 > n^2 - n$ points and constant point degree $n + 1$. By Lemma 2.3, $\boldsymbol{L}''$ can be embedded into a projective plane $\boldsymbol{P}$ of order n. Since $\boldsymbol{L}'$ is embedded in $\boldsymbol{L}''$, this proves Lemma 3.5. □

Now we come to the end of the proof of Theorem 2.

Let p_1 and p_2 be two points of the line H, and denote by Π_i, $i = 1, 2$, the set of the $n + 1$ lines other than H which contain p_i. Obviously Π_1 and Π_2 are parallel classes of $\boldsymbol{L}'$ which have no line in common.

By Lemma 3.5, $\boldsymbol{L}'$ has at least $n^2 - n$ points and $\boldsymbol{L}'$ can be embedded into a projective plane $\boldsymbol{P}$ of order n. Lemma 2.1(b) shows that the lines of Π_i, $i = 1, 2$, intersect in $\boldsymbol{P}$ in a point q_i. In view of $|\Pi_i| = n + 1$, the line $q_1 q_2$ of $\boldsymbol{P}$ is contained in Π_i. This contradicts $\Pi_1 \cap \Pi_2 = \emptyset$.

This final contradiction shows that every point other than p_0 has degree $n + 1$ so that Theorem 2 follows from Lemma 3.2. □

4. Proof of Theorem 3

Throughout this section, $\boldsymbol{L}$ denotes a linear space satisfying the conditions of Theorem 3, and n denotes the integer with $n^2 - \frac{1}{2}n + 1 < v \leqslant b \leqslant n^2 + n + 1$. We shall prepare the proof of Theorem 3 in several lemmas for which we shall use the following notation:

$$s = n^2 + 1 - v, \qquad T = \{p \mid p \text{ is a point with } r_p \geqslant n + 2\},$$

$$t = \sum_{p \in T} (r_p - n - 1), \qquad v' = v - |T|,$$

$$t_L = \sum_{p \in L} (r_p - n - 1) \quad \text{and} \quad d_L = n + 1 - k_L \text{ for every line } L \text{ of } \boldsymbol{L}.$$

Lemma 4.1. *Every point has degree at least $n + 1$, and every line has degree at most n.*

Proof. Because $\boldsymbol{L}$ has a point of degree at least $n + 2$, $\boldsymbol{L}$ can not be embedded into a projective plane of order n. Since $\boldsymbol{L}$ is also not a near pencil, Theorem 2 shows that every point of $\boldsymbol{L}$ has degree at least $n + 1$.

Assume, a line L has degree at least $n + 1$. Because every point of L has degree at least $n + 1$, L intersects at least $k_L n$ lines. In view of $b \leqslant n^2 + n + 1$, this shows that $k_L = n + 1$, that L intersects every line and that every point of L has degree $n + 1$. Hence, if p is a point of degree at least $n + 2$, then $p \notin L$. Consequently, p is contained in $r_p - k_i \geqslant 1$ lines parallel to L, a contradiction. □

Lemma 4.2. *There exists a point of degree $n+1$.*

Proof. Assume to the contrary that there is no point of degree $n+1$. Then Lemma 4.1 shows that every point has degree at least $n+2$. Because every line has degree at most n, this implies $v(n+2) \leqslant bn$, a contradiction. □

Lemma 4.3. *If a line L contains a point of degree $n+1$, then $t_L \leqslant d_L \leqslant s$ and L contains at least two points of degree $n+1$.*

Proof. Let p be a point of degree $n+1$ contained in L. Because every line has degree at most n and in view of $v > n^2 - n + 1$, p is contained in at least two lines of degree n. Thus, p is contained in a line N of degree n with $N \neq L$. Obviously, L intersects exactly $k_L - 1 + t_L = n - d_L + t_L$ of the lines which are parallel to N. Since N is parallel to $b - 1 - k_N n - t_N \leqslant n$ lines, this shows $t_L \leqslant d_L$. Because every line of L has degree at most n, we have furthermore

$$n^2 + 1 - s = v \leqslant k_L + (r_p - 1)(n-1) = n^2 + 1 - d_L.$$

Consequently $t_L \leqslant d_L \leqslant s$ and $k_L - t_L = n + 1 - d_L - t_L \geqslant n + 1 - 2s > 1$ so that L contains at least two points of degree $n+1$. □

Lemma 4.4. *Let L be any line. Then there is a line N parallel to L such that $(d_L n - t_L)k_N \geqslant (v - k_L)d_L$.*

Proof. L is parallel to $b - 1 - k_L n - t_L \leqslant d_L n - t_L$ lines. Consequently, if N is a line of maximal degree of the lines parallel to L, then

$$(v - k_L)d_L \leqslant \sum_{p \notin L} (r_p - k_L) = \sum_{X \cap L = \emptyset} k_X \leqslant (d_L n - t_L)k_N. \quad \square$$

Lemma 4.5. *Let G and L be two different intersecting lines. If there is a line N of degree n which is parallel to G and L, then $d_G d_L \geqslant n - t_N$.*

Proof. Let p be a point on G which is not the intersecting point of G and L. Then p is contained in $r_p - n$ lines parallel to N and in $r_p - k_L$ lines parallel to L. Since G is parallel to N but not to L, it follows that p is contained in at least $r_p - k_L - (r_p - n - 1) = d_L$ lines parallel to L but not to N. Consequently, G intersects at least $(k_G - 1)d_L$ of the lines which are parallel to L but not to N. Since N intersects exactly $k_N(d_L - 1) + t_N$ lines which are parallel to L, it follows $(k_G - 1)d_L \leqslant n(d_L - 1) + t_N$. Consequently $d_G d_L \geqslant n - t_N$. □

Lemma 4.6. *$t_N = 0$ for every line N of degree n.*

Proof. Let L be any line of degree n. Lemma 4.4 shows $t_L < n$. Consequently, L contains a point of degree $n+1$ so that $t_L \leqslant 1$ by Lemma 4.3.

Assume by way of contradiction that $t_L = 1$. Then L contains a unique point q of degree $n+2$. By Lemma 4.4, L is parallel to a line N of degree n. As before, $t_N \leq 1$. Let $G \neq L$ be the second line which contains q and which is parallel to N. Lemma 4.5 shows $d_G = n-1$ and $t_N = 1$. Since N is parallel to $b-1-k_N n - t_N \leq n-1$ lines, it follows

$$v - n \leq \sum_{p \notin N} (r_p - k_N) = \sum_{X \cap N = \emptyset} k_X \leq k_G + (n-2)n = n^2 - 2n + 2.$$

This contradicts $v > n^2 - \frac{1}{2}n + 1$ (notice $n = k_N \geq 2$). □

Lemma 4.7. *Every point of T has degree at least $n+3$.*

Proof. Assume by way of contradiction, a point p of T has degree $n+2$. In view of Lemma 4.6, p is not contained in a line of degree n. Since $v-1 > r_p(n-3)$, this shows that p is contained in a line L of degree $n-1$. Because p has degree $n+2$, we have $t_L \geq 1$. Let N be a line of degree n which is parallel to L (Lemma 4.4), and denote by G the unique line $\neq L$ which contains p and which is parallel to N. By Lemma 4.5 and Lemma 4.6, $k_G \leq \frac{1}{2}n + 1$. It follows

$$v \leq k_G + (r_p - 1)(n-2) = k_G + n^2 - n - 2 \leq n^2 - \tfrac{1}{2}n - 1,$$

a contradiction. □

Lemma 4.8. $|T| \leq s - 1$ *and* $v' > n^2 - n + 1$.

Proof. Let p be a point of degree $n+1$ (Lemma 4.2), denote by e the number of lines of degree n which contain p, and set $f = n + 1 - e$. If $L_1, \ldots, L_f$ are the lines of degree at most $n-1$ which contain p, then

$$\begin{aligned} n^2 - s = v - 1 = e(n-1) + \sum_{i=1}^{f} (k_{L_i} - 1) &\leq e(n-1) + f(n-2) \\ &= n^2 - n - 2 + e. \end{aligned}$$

Hence $e \geq n + 2 - s$. In view of $k_{L_i} - 1 = n - d_{L_i}$, we obtain furthermore

$$\sum_{i=1}^{f} d_{L_i} = e(n-1) + fn + s - n^2 = n + s - e \leq 2s - 2.$$

Together with Lemma 4.3 and Lemma 4.6 follows

$$t = \sum_{q \in T} (r_q - n - 1) = \sum_{i=1}^{f} t_{L_i} \leq \sum_{i=1}^{f} d_{L_i} \leq 2s - 2.$$

By Lemma 4.7, $|T| \leq \frac{1}{2}t$. Hence $|T| \leq s - 1$ and $v' = v - |T| > n^2 - n + 1$. □

Lemma 4.9. *Every point of T has degree at least $n+4$.*

Proof. Assume to the contrary, some point p of T has degree $n+3$ (Lemma 4.7). We consider two cases.

Case 1. p is not contained in a line of degree $n-1$

Since p is also not contained in a line of degree n (Lemma 4.6) and in view of $v-1>r_p(n-4)$, p lies on a line L of degree $n-2$. Because p has degree $n+3$, we have $t_L\geqslant 2$. Let N be a line of degree n which is parallel to L (Lemma 4.4), and let G and G' be the two lines $\neq L$ which are parallel to N and which pass through p. Since every line which contains p has degree at most $n-2$, we have

$$n^2-\tfrac{1}{2}n<v-1\leqslant(k_G-1)+(k_{G'}-1)+(r_p-2)(n-3)=n^2-3-d_G-d_{G'}.$$

This contradicts Lemma 4.5 and Lemma 4.6.

Case 2. p is contained in a line L of degree $n-1$

As in Case 1, L is parallel to a line N of degree n, and p is contained in two lines G and G' different from L which are parallel to N. Lemma 4.5 shows $k_G\leqslant\tfrac{1}{2}n+1$. Therefore it is easy to see that G does not contain a point of degree $n+1$. Hence, every point of G is a point of T. Because the same holds for G', Lemma 2.8 shows that $k_G+k_{G'}\leqslant s$. On the other hand, since N is parallel to at most n lines and since every point outside of N is contained in at least one of the lies which are parallel to N, we have

$$n^2+1-s-n=v-k_N\leqslant k_G+k_{G'}+(n-2)n.$$

Consequently $k_G+k_{G'}\geqslant n+1-s>s$, a contradiction. □

Lemma 4.10. *Let p be a point of degree $n+1$, and denote by e the number of lines of degree n containing p. Then $v'+e>n^2$.*

Proof. Let f be the number of lines of degree $n-1$ through p, set $g=n+1-e-f$, and denote by $L_1,\ldots,L_g$ the lines of degree at most $n-2$ through p. Then

$$n^2-s=v-1=e(n-1)+f(n-2)+\sum_{i=1}^{g}(k_{L_i}-1).$$

In view of $k_{L_i}-1\leqslant n-3$, we obtain

$$2e+f\geqslant n^2-s-(e+f+g)(n-3)=2n+3-s.$$

Furthermore, since $k_{L_i}-1=n-d_{L_i}$, we get

$$\sum_{i=1}^{g}d_{L_i}=(e+f+g)n-e-2f+s-n^2=n+s-e-2f.$$

Because no point of T lies on a line of degree n (Lemma 4.6) or on a line of degree $n-1$ through p (Lemma 4.3 and Lemma 4.9), we have

$$t=\sum_{q\in T}(r_q-n-1)=\sum_{i=1}^{g}t_{L_i}.$$

In view of Lemma 4.3, it follows

$$t\leqslant n+s-e-2f.$$

Lemma 4.9 shows $3|T|\leqslant t$. Thus

$$\begin{aligned} v'+e=v-|T|+e &\geqslant n^2+1-s-\tfrac{1}{3}t+e \\ &\geqslant n^2+1-s-\tfrac{1}{3}(n+s-e-2f)+e \\ &= n^2+1-\tfrac{1}{3}n-\tfrac{4}{3}s+\tfrac{2}{3}(2e+f) \\ &\geqslant n^2+1-\tfrac{1}{3}n-\tfrac{4}{3}s+\tfrac{2}{3}(2n+3-s) \\ &= n^2+3+n-2s>n^2. \qquad \square \end{aligned}$$

From now on, let m be the unique positive integer with $(m-1)^2<n\leqslant m^2$.

Lemma 4.11. *A line L with $t_L\geqslant m$ has degree at most $n+1-m$.*

Proof. If L contains a point of degree $n+1$, this follows from Lemma 4.3. If no point of L has degree $n+1$, then $L\subseteq T$ so that $k_L\leqslant|T|\leqslant s-1\leqslant n+1-m$ by Lemma 4.8. □

Lemma 4.12. *T contains at most one point p with $r_p\geqslant n+1+m$.*

Proof. Assume to the contrary that T contains two points p and q of degree at least $n+1+m$. Let N be a line of degree n (such a line exists, since every point of degree $n+1$ is contained in a line of degree n). Then p and q are not on N (Lemma 4.6). Hence, N is parallel to at least $2m+1$ lines which contain p or q. Each of these lines has degree at most $n+1-m$ (Lemma 4.11). Since N is parallel to at most n lines, it follows

$$\begin{aligned} &n^2+1-s-n+2m \\ &\quad\leqslant\sum_{x\notin N}(r_x-k_N)=\sum_{X\cap N=\emptyset}k_X \\ &\qquad\qquad\leqslant(2m+1)(n+1-m)+(n-2m-1)n \\ &\qquad\qquad=n^2-2m^2+m+1\leqslant n^2-2n+m+1. \end{aligned}$$

This contradicts $s<\frac{1}{2}n$. □

Now we are ready for the proof of Theorem 3.

By Lemma 4.3, every line, which is not contained in T, contains at least two points of degree $n+1$. Thus, the points of degree $n+1$ together with the lines not contained in T form a linear space L' with constant point degree $n+1$. L' has $v'>n^2-n+1$ points (Lemma 4.8) and at most b lines. Furthermore, every line, which has degree n in L, has still degree n in L' (Lemma 4.6). Now, Result 1 and Lemma 4.10 show that L' can be embedded into a projective plane P of order n.

By the hypotheses of Theorem 3, L has a point p of degree at least $n+2$. By definition, $p \in T$. Let Π be the set of lines of L', which contain p in L. Then Π is a parallel class of L'. Furthermore, every line of Π has degree at most $n-2$ in L', since a line of degree n of L does not contain a point of T. Hence, $v' \leqslant |\Pi|(n-2)$ and thus $|\Pi| \geqslant n+2$. Lemma 2.1 shows $|\Pi| \geqslant n+1+m$. In particular, p has degree at least $n+1+m$. In the same way follows that every point of T has degree at least $n+1+m$. Lemma 4.12 yields $T=\{p\}$. Consequently, $v'=v-1>n^2-\frac{1}{2}n$. Let B be the set of points of P, which lie on at least two of the lines of Π. Lemma 2.2 shows that (B, Π) is a Baer-subplane of P. Obviously, L' is embedded in $P-B$ and any two distinct lines of Π are parallel in $P-B$. Consequently, L can be embedded into the closed complement of (B, Π) in P. This completes the proof of Theorem 3. □

Proof of the corollaries. Corollary 1 follows from Theorem 2, Theorem 3, Lemma 2.3 and Result 2. Corollary 2 and Corollary 3 are immediate consequences of Corollary 1.

Corollary 2 implies $A(n) \leqslant n^2$ for all n. Since the near pencil on three points is the only linear space with three points, it follows $A(1)=A(2)=1$. The remark following the proof of Theorem 1 shows $A(3) \geqslant 9$ so that $A(3)=9$. $A(4)=16$, by Corollary 3. □

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